8. Since $\frac{n}{(999\ldots9)_r}$ is the general form of a common fraction producing a repetend of r figures, it follows that any divisor of $(999\ldots9)_r$ is a divisor of a number if it is a divisor of the sum of its digits taken r together.

NOTE. In a letter to the Editor, Prof. H. T. Eddy writes:—"In the very interesting Historical Sketch contained in your Sept. No. there is one omission which I feel should be supplied in the enumeration of the articles contributed to the Mathematical Monthly. I refer to Ferrel's investigations respecting the laws which govern atmospheric currents. These articles are regarded, I think, as the most important original investigations published in the Mathematical Monthly."

SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. II.

SOLUTIONS of problems in No. 6, Vol. II, have been received as follows: From J. M. Arnold, 92; Prof. W. W. Beman, 93 and 94; Lieut. S. H. Baker U. S. N., 94; G. L. Dake, 92 and 95; G. M. Day, 97; Cadet E. S. Farrow, 92, 93, 94, 95 and 97; Henry Gunder, 92, 95 and 97; Christine Ladd, 93, 94 and 95; Artemas Martin, 92 and 95; Dr. A. B. Nelson, 92 and 95; O. D. Oathout, 92, and 97; Geo. H. Pegram, 95; K. M. Supten, 92 and 95; Prof. J. Scheffer, 92, 93, 94, 95 and 97; E. B. Seitz, 95 and 97; E. H. Westermann, 95; Prof. C. M· Woodward, 97.

92.—"A balloon is ascending vertically with a given velocity v, and a body is let fall from it, which touches the ground in t seconds; find the height of the balloon at the moment the body is let fall from it."

SOLUTION BY HENRY GUNDER, NORTH MANCHESTR, IND.

In t seconds a body will fall from *rest* $\frac{1}{2}gt^2$ feet. But from the conditions of the problem it ascends vt feet. Therefore it falls from a height of $(\frac{1}{2}gt—v)t$ feet.

93.—"To construct a triangle if the three radii of the circles, which touch the three sides externally, are given."

SOLUTION FURNISHED BY PROF. W. W. BEMAN, ANN ARBOR, MICH.

Christine Ladd writes: "The construction follows at once from the solution given by Chauvenet, 164, 12." Prof. Beman writes: "The following elegant solution of Problem 93 may be found – *in substance* – in Catalan's '*Theoremes et Problemes de Geometrie Elementaire*', page 155:

Let r, r', and r'' be the radii of the escribed circles, a, b, and c, the corresponding sides of the triangle.

Equating three expressions for *twice the area* of the triangle, we have

$$r(b+c-a) = r'(c+a-b) = r''(a+b-c).$$

$$\therefore \qquad \frac{b+c-a}{r'} = \frac{c+a-b}{r} = \frac{a+b-c}{rr' \div r''},$$

whence we easily obtain

$$\frac{a}{r + rr' \div r''} = \frac{b}{r' + rr' \div r''} = \frac{c}{r + r'}.$$

The triangle required is therefore similar to the one whose sides are $r + rr' \div r''$, $r' + rr' \div r''$, and $r + r'$.

Construct this triangle, draw its escribed circles, and through points at distances from their centres equal to r, r', and r'', draw parallels to its sides. We then have the triangle."

94.—"One side of a quadrilateral, whose four sides are given in length, is fixed: find the equation of the locus of the middle point of the opposite side, in rectangular coordinates."

SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.

We have the following equations from which to determine the relation between x and y, if the origin is at one extremity of the fixd side.

$$x = \tfrac{1}{2}(x' + x''), \qquad y = \tfrac{1}{2}(y' + y''),$$

$$x'^2 + y'^2 = d^2,\ (x'' - a)^2 + y''^2 = b^2,\ (x'' - x')^2 + (y'' - y')^2 = c^2.$$

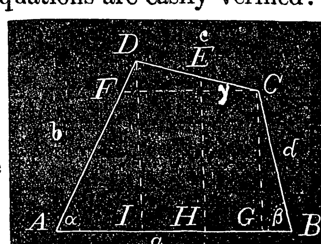
On elimination an equation of the sixth degree is obtained. When the quadrilateral is a rhombus and the origin is at the middle point of the fixed side, whose length is $2a$, the equation reduces to

$$x^6 + 3x^4y^2 + 3x^2y^4 - 8a^2x^4 - 12a^2x^2y^2 + 16a^4x^2 - 4a^2y^4 + y^6 = 0.$$

This is the equation of three circles with radii respectively equal to a, a $2a$, and centres at $(a, 0)$ $(—a, 0)$ and $(0, 0)$.

SOLUTION BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD.

By referring to the diagram the following equations are easily verified:



$$b \sin \alpha + d \sin \beta = 2y, \quad . \ . \ . \ . \quad (1)$$

$$b \cos \alpha + \tfrac{1}{2} c \cos \gamma = x, \quad . \ . \ . \ . \quad (2)$$

$$d \cos \beta + \tfrac{1}{2} c \cos \gamma = a - x, \quad . \ . \quad (3)$$

$$b \sin \alpha - d \operatorname{ssn} \beta = c \sin \gamma. \quad . \ . \quad (4)$$

By combining (1) and (4), and (2) and (3) we get

$$\sin \alpha = \frac{2y + c \sin \gamma}{2b}, \quad . \ . \ . \ . \quad (5)$$

$\sin\beta = \dfrac{2y - c\sin\gamma}{2d}$ (6), $\cos\alpha = \dfrac{2x - c\cos\gamma}{2b}$ (7), $\cos\beta = \dfrac{2(a-x) - c\cos\gamma}{2d}$. (8)

Squaring and adding (5) and (7), and also (6) and (8) we have,

$$8y^2 + 2c^2 + 4x^2 + 4(a - x)^2 - 4ac\cos\gamma = 4b^2 + 4d^2, \text{ whence}$$

$$\cos\gamma = \frac{4y^2 + 4x^2 - 4ax + c^2 - 4b^2 - 4d^2}{2ac}.$$

Substituting this value of $\cos\gamma$ in (7) and (8) we obtain

$$\cos\alpha = \frac{8ax - 4y^2 - 4x^2 - c^2 + 4b^2 + 4d^2}{4ab},$$

$$\cos\beta = \frac{4a^2 - 4y^2 - 4x^2 - c^2 + 4b^2 + 4d^2}{4ad}.$$

Adding (2) and (3) we have $\quad b\cos\alpha + d\cos\beta + c\cos\gamma = a.$

Substituting in this equation for $\cos\alpha$, $\cos\beta$ and $\cos\gamma$, their values as found above we obtain for the equation of the locus,

$$4y^2 + 4x^2 - 4ax = 4b^2 + 4d^2 - 2a^2 - c^2,$$

or

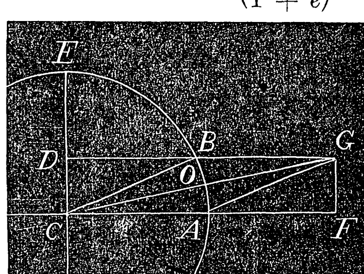
$$y^2 + (x - \tfrac{1}{2}a)^2 = b^2 + d^2 - \tfrac{1}{4}(a^2 + c^2).$$

Consequently the locus is a circle, whose centre is the middle point of the side AB, and whose radius $= \sqrt{[b^2 + d^2 - \tfrac{1}{4}(a^2 + c^2)]}$.

[Prof. Beman has given the equation of the locus of *any* point in the line DC. His equation is of the 6th degree. Prof Woodward writes: "Prob. 94, is a special case of the well-known general problem connected with Watt's parallel motion. The full solution, with careful discussion is given by Prony in his *Architecture Hydraulique*."]

95.—"Prove, otherwise than by the Integral Calculus, that

$$\frac{\pi}{2} - \sin^{-1}e = 2\tan^{-1}\left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}}."$$



Solution by Dr. Nelson.—Let $e = DB = \sin EB$. Then $AB = \tfrac{1}{2}\pi - \sin^{-1}e$. Make $AF = DB$ and complete the rectangle $CFGD$. Then, radius being unity, $CF = 1 + e$, $GF = (1 - e^2)^{½}$, and $\tan OA = \dfrac{(1-e^2)^{½}}{1+e} = \left(\dfrac{1-e}{1+e}\right)^{\frac{1}{2}}$. We have to show that $AB = 2OA$. Draw CB and AG. $CAGB$ is easily seen to be a rhombus; hence CG bisects ACB, & $AB = 2AO$.

96.—"A plays m games with B whose skill is equal to his own. Required the probability that one of them will win n consecutive games."

[Several solutions of this question have been received, but in all of them the probability of winning n consecutive games merely is considered, and no account is taken of the fact that there are m games in which the *run* may occur. If m is greater than n, as is presumably the case in this question, the chance of a run of n games is obviously greater than if n games only were played.

When m is a small number it is not difficult to determine the required chance, as all the possible results may be presented to the eye, and the runs of n games counted, thus:

If $m = 6$ and $n = 2$, the number of possible different events is $2^6 = 64$, by writing down which, and counting, we find, in the 64 sets of 8 games each which represent all the possible events, 43 in which a *run* of two or more occurs. Hence the chance that either one of the players will win two consecutive games in a set of six games is $\frac{43}{64}$, or more than one-half.

When m is a large number, however, this method becomes impracticable, and the solution involves a somewhat intricate analysis. De Morgan has given the solution of an analogous question as an example of the application of Laplace's Theory of Generating Functions. He gets, as a representative of the required probability, the following formula:

$$p^n[p'(m-n)+1] - p^{2n}p'\,\frac{m-2n}{1.2}\Big[p'(m-2n-1)+2\Big]$$
$$+\,p^{3n}p'^2\,\frac{(m-3n-1)(m-3n)}{1.2.3}\Big[p'(m-3n-2)+3\Big] - \&c$$

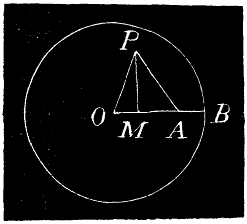
In this formula p is the probability of a single event, $= \frac{1}{2}$ in this problem, and $p' = 1 - p$.—Ed.]

97.—"Find the average distance of all the points of a sphere, radius r, from a point whose distance from the center is a."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

There are two cases. The given point may be within the sphere, or without the sphere.

1. Let A be the given point, O the center, and P any point of the sphere. Draw PM perpendicular to AO.



Put $OB = r$, $OA = a$, $OM = x$, $PM = y$. Then $AP = [(a-x)^2 + y^2]^{\frac{1}{2}}$, when $x < a$, and $AP = [(x-a)^2 + y^2]^{\frac{1}{2}}$, when $x > a$; hence the sum of the distances of all the points of the sphere from A is

$$\int_{-r}^{a}\int_{0}^{u} 2\pi y[(a-x)^2+y^2]^{\frac{1}{2}}dydx + \int_{a}^{r}\int_{0}^{u} 2\pi y[(x-a)^2+y^2]^{\frac{1}{2}}dydx$$

(in which the superior limit $u = \sqrt{(r^2-x^2)}$)

$$=\tfrac{2}{3}\pi\int_{-r}^{a}[(a^2+r^2-2ax)^{\frac{3}{2}}-(a-x)^3]dx + \tfrac{2}{3}\pi\int_{a}^{r}[(a^2+r^2-2ax)^{\frac{3}{2}}-(x-a)^3]dx$$

$$=\tfrac{2}{3}\pi\left\{\frac{1}{5a}[(r+a)^5-(r-a)^5]-\tfrac{1}{4}[(r+a)^4+(r-a)^4]\right\}$$

$$=\tfrac{4}{3}\pi r^3\left(\frac{3r}{4}+\frac{a^2}{2r}-\frac{a^4}{20r^3}\right).$$

Hence the average distance is

$$\tfrac{4}{3}\pi r^3\left(\frac{3r}{4}+\frac{a^2}{2r}-\frac{a^4}{20r^3}\right)\div\tfrac{4}{3}\pi r^3=\frac{3r}{4}+\frac{a^2}{2r}-\frac{a^4}{20r^3}.$$

2. When the given point is without the sphere, the sum of the distances of all the points of the sphere from A is

$$\int_{-r}^{r}\int_{0}^{u} 2\pi y[(a-x)^2+y^2]^{\frac{1}{2}}dydx = \tfrac{2}{3}\pi\int_{-r}^{r}[(a^2+r^2-2ax)^{\frac{3}{2}}-(a-x)^3]dx$$

$$=\tfrac{2}{3}\pi\left\{\frac{1}{5a}[(a+r)^5-(a-r)^5]-\tfrac{1}{4}[(a+r)^4-(a-r)^4]\right\}=\tfrac{4}{3}\pi r^3\left(a+\frac{r^2}{5a}\right).$$

Hence the average distance is $\tfrac{4}{3}\pi r^3\left(a+\frac{r^2}{5a}\right)\div\tfrac{4}{3}\pi r^3 = a+\frac{r^2}{5a}$. When $a = r$, the above results both reduce to $\tfrac{6}{5}r$.

The "QUERY" in No. 6, Vol. II, was answered by E. B. Seitz, E. S. Farrow, Prof. J. Scheffer, Prof. A. B. Evans, Prof. A. Hall, and by Prof. H. T. Eddy. All the solutions are elegant, and most of them brief. The following is by Prof. Eddy of Cincinnati, Ohio.

$$u = \int_0^{\frac{\pi}{2}}\frac{x\sin x.dx}{(1-p^2\sin^2 x)^{\frac{3}{2}}}.\qquad\text{By parts,}$$

$$u = \frac{-1}{1-p^2}\left[\frac{x\cos x}{(1-p^2\sin^2 x)^{\frac{1}{2}}}-\int\frac{\cos x.dx}{(1-p^2\sin^2 x)^{\frac{1}{2}}}\right]_0^{\frac{\pi}{2}}$$

$$\therefore\quad u = \frac{-1}{1-p^2}\left[\frac{x\cos x}{(1-p^2\sin^2 x)^{\frac{1}{2}}}-\frac{1}{p}\sin^{-1}(p\sin. x)\right]_0^{\frac{\pi}{2}}$$

$$\therefore\quad u = \frac{\sin^{-1}p}{p(1-p^2)}.$$

[After the foregoing solutions were put in type we received, from J. M. Greenwood and W. H. Baker, solutions of 92, 93. 95, 96 and 97. The following is the formula they obtain for the solution of 96:

$$\left(\frac{1}{2}\right)^m\left(r+1+r^2+\frac{r(r-1)^2}{1.2}+\frac{r(r-1)(r-2)^2}{1.2.3}\ldots+\frac{\underline{|r}}{\underline{|r}}\right).$$

In this formula $m-n=r$, and $n>r$.]